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Kinematics of the Forced and Overdamped Sine-Gordon Soliton Gas

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Motion of a driven and heavily damped sine-Gordon chain with a low density of kinks and tight coupling between particles is controlled by the nucleation and subsequent annihilation of pairs of kinks and antikinks. We show that in the steady state there are no spatial correlations between kinks or between kinks and antikinks. For a given number of kinks and antikinks all geometrical distributions are equally alike, as in equilibrium. A master equation for the probability distribution for the number of kinks on a finite chain is solved, and substantiates the physical reasoning in previous work. The probability distribution characterizing the spread along the direction of particle motion of a finite chain in equilibrium as well as in the driven overdamped case is derived by simple combinatorial considerations. The spatial spread of a driven chain in the thermodynamic limit does not approach a steady state; a given particle followed in time deviates as $t^{1/2}$ from its average forced motion. This result follows from the hydrodynamic equations for the dilute kink gas. Comparison is made with other recent results.

KEY WORDS: Sine-Gordon soliton gas; nucleation; annihilation; master equation; hydrodynamic equations; fluctuations; correlations.

1. INTRODUCTION

In this paper we study the statistical properties of the kink–antikink gas of the sine-Gordon equation, with emphasis on the forced and overdamped case, though some of our results are more general. One possible realization consists of a ring of torsion-coupled pendulums in a gravitational potential $V(1 - \cos\theta)$, and also under the action of an external torque F. In the

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overdamped case the time evolution of the displacement $\theta(x,t)$ of the pendulums is given by[1,2]

$$\gamma\,\partial\theta/\partial t = -V\sin\theta + F + \kappa\,\partial^2\theta/\partial x^2 + \zeta \tag{1}$$

where γ is the damping constant and κ is proportional to the coupling between adjacent pendulums. The system is connected to a thermal reservoir giving rise to fluctuations with a strength

$$\langle \zeta(x,t)\zeta(x',t')\rangle = 2\gamma kT\,\delta(t-t')\,\delta(x-x') \tag{2}$$

Equations (1) and (2) describe other physical systems,[3,4] such as the Josephson junction transmission line[5,6] with negligible junction capacitance, or a chain of oscillators with phase coupling between adjacent oscillators and synchronized externally by a signal differing from the natural oscillator frequency.[7] Recent conference proceedings[3,4] demonstrate many other physical applications of the sine-Gordon equation, though for many of these the inertial term may not be neglected as we have done in Eq. (1).

For $|F| < V$ the potential $V(1-\cos\theta) - F\theta$ possesses local minima at $\theta_s + 2\pi n$. At low temperatures the statistical mechanical properties in this field range are determined by two elementary types of excitations[8]: small-amplitude relaxation modes ("overdamped phonons") around the stationary uniform states θ_s, and large-amplitude excitations (kinks and antikinks) describing the transition from one Peierls valley at θ_s to an adjacent one at $\theta_s \pm 2\pi$. We adopt the notions "Peierls valley" and "Peierls hill" from the dislocation literature,[8] the field in which the statistical mechanics of solitons was treated first.

We will call a transition from one Peierls valley to an adjacent one a kink if the first spatial derivative is positive, and an antikink if the first spatial derivative is negative. As a linguistic simplification, we will, on occasion, use the expression "kink" as a generic term, including both types of transitions. Under the action of a steady deterministic field, a kink travels with constant velocity[2] $-u(F)$ in the presence of damping (Fig. 1), and an antikink travels with velocity $u(F)$ to the right. In the absence of a force the kink is at rest. (In contrast the kinks of the undamped sine-Gordon equation can travel with any fixed velocity smaller than the velocity of sound. The velocity for $F=0$, $\gamma=0$ depends on the order in which the limits $F\to 0$, $\gamma\to 0$ are taken.)

In this paper we study the statistical properties of a dilute kink gas. In the overdamped case and in the low-temperature regime, the kinematics of this gas is governed by two processes. If a kink and an antikink collide, they will annihilate (recombine). There is an attractive force between them; and since excess kinetic energy is immediately removed by the damping,

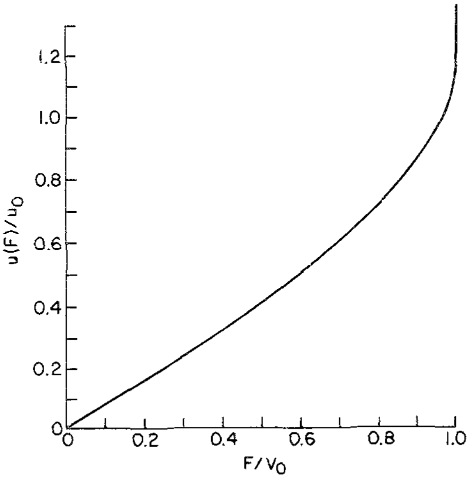

Fig. 1. Propagation velocity of the driven kinks. A kink travels with velocity $-u$, i.e., toward negative x values; an antikink travels with velocity u toward positive x values. In the reduced variables u/u_0, F/V the curve shown is universal. $u_0 = (\kappa V)^{1/2}/\gamma$. Corrected version of figure in Ref. 2.

they cannot separate again. The complementary process to the recombination is the nucleation of a pair, consisting of a kink and an antikink.(2,8) Fluctuations out of the uniform state θ_s which are large enough to overcome a critical activation energy barrier(2) will lead to a new kink–antikink pair which, under the action of the applied field, are then driven apart. Throughout the paper, we assume that kT is small compared to the activation energy barrier.

We treat the kinks as classical point-like particles interacting only via short-range forces. Because the interaction between kinks decays exponentially over distances larger than the kink width,(8,9) such a treatment is appropriate if one considers a length scale which is large compared to the width of a kink.

In Section 2, we show that there are no spatial correlations between kinks and antikinks in the steady state. It is found that all geometrical distributions of kinks and antikinks are equally likely. We derive and solve a master equation for the probability distribution for the total number of kinks. We discuss the validity of balance equations which have been invoked(2) in the determination of the mean angular velocity $\langle \partial\theta(F)/\partial t \rangle$ for the system of Eqs. (1) and (2). The determination of this velocity is equivalent to the determination of crystal growth rates(10,11) for a one-dimensional crystal surface. In Section 3, we derive hydrodynamic equations describing the driven and overdamped kink gas, and study the correlation functions derivable from these equations. In Section 4 we derive the probability distribution characterizing the spatial spread of the finite chain along the direction of particle motion. This section is more general than the preceding ones because it covers the equilibrium as well as the driven case, the overdamped as well as the underdamped case. In the last section, we discuss the relation of our results to earlier work.

2. NUCLEATION AND RECOMBINATION OF KINKS AND ANTIKINKS

2.1. Formulation of the Problem

Figure 2a illustrates the distribution of the angular displacement $\theta(x,t)$ of a typical member of the ensemble over the valleys of the potential $V(1-\cos\theta)-F\theta$ at a given instant of time. At low temperature the kinks are far apart compared to the width of a kink and most of the chain will lie near the potential minima at $\theta_s+2\pi n$. Segments of the chain lying in different Peierls valleys are connected by kinks and antikinks. The time evolution of this configuration occurs through motion of the kinks to the left with speed $-u(F)$ and motion of the antikinks to the right with speed

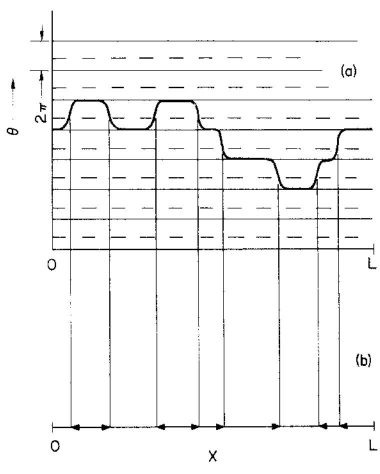

Fig. 2. (a) Typical configuration of the displacement field $\theta(x,t)$ at a given instant of time. Long segments of the chain lying in a Peierls valley (thin solid lines) are connected by kinks and antikinks which span the Peierls hills (broken solid lines). (b) One-dimensional representation of the configuration of Fig. 2a through the kink and antikink positions with arrows indicating the sign of their propagation velocity, in the presence of a force, $F>0$, leading to positive values of the ensemble average of $\partial\theta/\partial t$.

$u(F)$. Noise will cause Brownian motion of the kinks away from the deterministic path described by $u(F)$. We are concerned with the long-term and large-amplitude displacements, and therefore for sufficiently large F the diffusive motion of the kinks can be ignored. The deviation from the deterministic path must still be small when the kink is annihilated at its next encounter with an antikink. The condition for this will be discussed in Section 3.

To describe the kinematics of such a kink gas, Seeger and Schiller[8] proposed balance equations describing the birth and death of kinks specifying separately the kink population spanning every Peierls hill. This leads to an infinite set of coupled nonlinear rate equations, which would be tractable only numerically on a computer. As we shall see, however, such a detailed description turns out not to be necessary. If one considers the kinks as point-like particles interacting only via short-range forces, each configuration can be represented in a one-dimensional space (Fig. 2b). The mapping from Fig. 2b to Fig. 2a is one to one. On a length scale large compared to the kink width both figures contain the same information.

We will show in this section that if we consider all the kinks and antikinks simultaneously, without any attempt to classify them by the Peierls hills they span, they exhibit no correlation in the steady state; all geometrical arrangements are equally likely. This simplicity is lost if we consider only the kinks and antikinks spanning a given Peierls hill. This gives rise to the complex equations discussed by Seeger and Schiller.[8] A particular antikink, for example, will find a kink spanning the same hill, and capable of annihilating it, with high probability in its immediate vicinity. Farther away, however, the chain will very likely have "diffused" away by a number of Peierls valleys, and the probability for finding a kink spanning the original hill is much less.

A configuration of an ensemble member consisting of N kinks and N antikinks distributed along a ring of length L is fully specified by the positions of the kinks x_r, $r = 1, \ldots, N$ and the positions of the antikinks y_s, $s = 1, \ldots, N$. Our main aim is to show that the stationary distribution $p(C_N)$ of the configurations

$$C_N = \{x_1, \ldots, x_N; y_1, \ldots, y_N\} \tag{3}$$

is uniform, e.g., independent of the values assigned to the coordinates $x_1, \ldots, x_N, y_1, \ldots, y_N$. We will search for a stationary distribution $p(C_N)$ of the form

$$p(C_N) = (1/V_N)p_N \tag{4}$$

where V_N is the volume of the phase space allowed for the configuration C_N, and p_N is the probability of finding N kinks in the system.

Why can we expect a uniform stationary distribution? Consider for a moment a gas consisting of N particles traveling with velocity $-u$ and N particles traveling with the velocity $+u$. Suppose these particles do not interact and pass through each other. One stationary distribution for this gas is the uniform distribution, giving equal weight to all possible spatial variations of the particle positions. If we do not invoke annihilation or nucleation, i.e., permit no transitions between the N-particle gas and the $(N+1)$-particle gas, then we can still assign these two classes any relative probability. Now let us, however, admit that we are dealing with kinks and allow transitions from the $(N+1)$-kink gas to the N-kink gas, and vice versa. Can we assign the relative overall populations p_N of Eq. (4) in such a way that the presumed uniform state specified in Eq. (4) will maintain itself in time? If we can find such a solution to the master equation it must be a unique solution, since the master equation allows nonunique steady states only when two or more parts of the phase space are mutually inaccessible, i.e., not connected by any sequence of allowed transitions.[12] Consider first the annihilation process. It requires that a kink and an antikink are about to run into each other. But, under the assumption of Eq. (4), this will occur with equal probability for all locations of the annihilation and for all possible assignments of the remaining N kinks and N antikinks, and will thus map the uniform distribution which existed on the space of $N+1$ kinks into the uniform distribution over the space of N kinks. Similarly the nucleation events taking us from the space with N kinks to that with $N+1$ occur with equal probability for all possible locations of the nucleation event, and with equal probability for all possible assignments of the N initial kinks and N initial antikinks. Thus the nucleation events predicted from Eq. (4) arising in the space of N kinks populate the space of $N+1$ kinks uniformly, except for the obvious restriction that the pair that has just been generated is just beginning to travel apart. Therefore, if we choose p_N and p_{N+1} correctly, the annihilation rate and the nucleation rate will not only balance after integration over the detailed geometrical assignments, but in a more microscopic way, so that the generation events appear as if on the average the kinks and antikinks had passed through each other without annihilation. Thus the uniform distribution assumed in Eq. (4) will be preserved.

We will now repeat in analytical form the material which has just been presented by verbal arguments. In Section 2.2, we derive the configuration-space master equation for the probabilities $p(C_N)$. In Section 2.3, we then deduce the master equation for the probabilities p_N and obtain its stationary solution. Finally, in Section 2.4 we show that the configuration-space master equation for $p(C_N)$ has a uniform stationary solution of the form (4) with p_N given by the stationary solution of the master equation for p_N.

2.2. Master Equation for the Probability of the Phase Space Configurations

Permutations of the kink positions x_r do not lead to new configurations C_N, nor do permutations of the antikink positions y_s. Therefore, the x_r and y_s may be considered ordered such that the allowed phase space is

$$\Omega_N = \{0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant L; \quad 0 \leqslant y_1 \leqslant y_2 \leqslant \cdots \leqslant y_N \leqslant L\} \tag{5}$$

with a volume

$$V_N = V(\Omega_N) = L^{2N}/(N!)^2 \tag{6}$$

Any integration over Ω_N may be replaced by an integration of the symmetrized integrand over L^{2N} and multiplication of the result by $(N!)^{-2}$,

$$\int_{\Omega_N} f(C_N)\, dx^N dy^N = \frac{1}{(N!)^2} \int_{L^{2N}} f_{\text{sym}}(C_N)\, dx^N dy^N \tag{7}$$

The nucleation affects the probability $p(C_N)$ for the configuration C_N in two ways: A nucleation event at $t = 0$ in the configuration C_{N-1} leads at $t = 0^+$ to a new configuration C_N with an extra pair of kink coordinates only slightly separated, $x_r - y_s = -0^+$. A nucleation event in the configuration C_N leads to a new configuration C_{N+1}. We obtain

$$\left.\frac{\partial p(C_N)}{\partial t}\right|_{\text{nuc}} = j \sum_{rs} p(C_N - \{x_r, y_s\})\, \delta(x_r - y_s + 0^+) - jLp(C_N) \tag{8}$$

where j is the nucleation rate per unit time and length calculated in Ref. 2.

In an interval dt a kink–antikink pair will annihilate if the pair members are within a distance $2u\,dt$. Recombinations (annihilations) in the configuration C_{N+1} will increase the probability for one of the configurations C_N, whereas annihilations in the configuration C_N will decrease this probability. Recombinations supply an additional contribution to $\partial p / \partial t$

$$\left.\frac{\partial p(C_N)}{\partial t}\right|_{\text{rec}} = 2u\Big[\int p(C_N + \{\hat{x}, \hat{y}\})\, \delta(\hat{x} - \hat{y} - 0^+)\, d\hat{x}\, d\hat{y} - \sum_{rs} p(C_N)\, \delta(x_r - y_s - 0^+)\Big] \tag{9}$$

The third process which we have to take into account is the deterministic motion of the kinks

$$\left.\frac{\partial p(C_N)}{\partial t}\right|_{\text{drift}} = +u \sum_r \left(\frac{\partial}{\partial x_r} - \frac{\partial}{\partial y_r}\right) p(C_N) \tag{10}$$

The evolution of the probability distribution is determined by all three processes

$$\left.\frac{\partial p(C_N)}{\partial t}\right|_{tot} = \left.\frac{\partial p(C_N)}{\partial t}\right|_{\text{nuc}} + \left.\frac{\partial p(C_N)}{\partial t}\right|_{\text{rec}} + \left.\frac{\partial p(C_N)}{\partial t}\right|_{\text{drift}} \tag{11}$$

2.3. Master Equation for p_N

We use Eq. (11) to derive a master equation for the probability distribution p_N containing as stochastic variable only the total number N of kinks (equal to the total number of antikinks) in our system. The distribution function p_N is found by integration of $p(C_N)$ over the phase space Ω_N

$$p_N = \int_{\Omega_N} p(C_N)\, dx^N dy^N = \frac{1}{(N!)^2} \int_{L^{2N}} p(C_N)\, dx^N dy^N \tag{12}$$

We find from (8)

$$\left.\frac{dp_N}{dt}\right|_{\text{nuc}} = jL(p_{N-1} - p_N) \tag{13}$$

and from (9)

$$\left.\frac{dp_N}{dt}\right|_{\text{rec}} = 2u\left[(N+1)^2 \langle\delta(x-y)\rangle_{N+1} p_{N+1} - N^2 \langle\delta(x-y)\rangle_N p_N\right] \tag{14}$$

whereas the deterministic motion leaves the distribution over N unchanged:

$$\left.\frac{dp_N}{dt}\right|_{\text{drift}} = 0 \tag{15}$$

Assuming at the moment a uniform distribution $p(C_N) = p_N / V_N$, with V_N defined by Eq. (6), we obtain

$$\begin{aligned}\langle\delta(x-y)\rangle_N &\equiv \frac{1}{N^2}\langle\sum_{rs} \delta(x_r - y_s)\rangle_N \\ &= \frac{1}{N^2}\frac{1}{p_N}\sum_{rs}\int dx^N dy^N\, p(C_N)\,\delta(x_r - y_s) = \frac{1}{L}\end{aligned} \tag{16}$$

and therefore for (14)

$$\left.\frac{dp_N}{dt}\right|_{\text{rec}} = \frac{2u}{L}\left[(N+1)^2 p_{N+1} - N^2 p_N\right] \tag{17}$$

Under the assumption of a uniform distribution $p(C_N)$, we thus find the macroscopic master equation

$$\frac{dp_N}{dt} = jL[p_{N-1} - p_N] + \frac{2u}{L}\left[(N+1)^2 p_{N+1} - N^2 p_N\right] \tag{18}$$

Its stationary solution is found by requiring that the nucleation rate is equal to the annihilation rate (detailed balance)

$$Ljp_N = (2u/L)(N+1)^2 p_{N+1} \tag{19}$$

and therefore

$$p_N = \frac{1}{(N!)^2}\left(\frac{jL^2}{2u}\right)^N p_0 \tag{20}$$

This solution is also found by considering the recursion relation obtained by setting the right-hand side of (18) equal to zero. Normalization of the distribution

$$1 = \sum_{N=0}^{\infty} p_N = p_0\left[\sum_{N=0}^{\infty} \frac{1}{(N!)^2}\left(\frac{jL^2}{2u}\right)^N\right] = p_0 I_0(x) \tag{21}$$

determimes p_0. Here $I_0(x)$ is the modified Bessel function of zeroth order with argument $x = (2jL^2/u)^{1/2}$. The characteristic function is found to be

$$\phi(q) = \sum_N e^{jqN} p_N = \frac{I_0(xe^{iq/2})}{I_0(x)} \tag{22}$$

and will be used to determine the moments of the stationary distribution.

2.4. The Stationary Uniform Solution of the Master Equation for $p(C_N)$

Now we show that $p(C_N) = p_N/V_N$ with p_N given by (20) is indeed a stationary solution of (11). Introducing the uniform distribution Eq. (4) into (13)–(15) yields

$$\left.\frac{\partial p(C_N)}{\partial t}\right|_{\text{nuc}} = j\frac{p_{N-1}}{V_{N-1}}\sum_{rs}\delta(x_r - y_s) - jL\frac{p_N}{V_N} \tag{23}$$

$$\left.\frac{\partial p(C_N)}{\partial t}\right|_{\text{rec}} = 2uL\frac{p_{N+1}}{V_{N+1}} - 2u\frac{p_N}{V_N}\sum_{rs}\delta(x_r - y_s) \tag{24}$$

and

$$\left.\frac{\partial p(C_N)}{\partial t}\right|_{\text{drift}} = 0 \tag{25}$$

whence

$$\begin{aligned}\frac{\partial p(C_N)}{\partial t} = {} & \frac{L^2}{(N+1)^2}\frac{1}{V_{N+1}}\left[\frac{2u}{L}(N+1)^2 p_{N+1} - jLp_N\right] \\ & - \frac{L}{N^2}\frac{1}{V_N}\left[\frac{2u}{L}N^2 p_N - jLp_{N-1}\right]\sum_{rs}\delta(x_r - y_s) = 0\end{aligned} \tag{26}$$

for p_N satisfying Eq. (19). Thus the master equation has indeed a uniform stationary distribution. It should be noted, however, that the second term on the right-hand side of (26) gives rise to a nonuniformity if p_N is not taken as the stationary distribution. In other words, the only distribution which is spatially uniform at all times is the stationary distribution. Thus, the macroscopic master equation (18) is strictly justified only for the stationary distribution, and any conclusions drawn from it concerning the approach to equilibrium should be taken with reservation.

2.5. Growth Rates

The average number of kinks in the stationary state is found with the help of Eq. (22) to be

$$\langle N \rangle = \frac{1}{i}\frac{\partial \phi}{\partial q} = \left(\frac{L^2 j}{2u}\right)^{1/2} \frac{I_1(x)}{I_0(x)} \tag{27}$$

where $I_1(x)$ is the modified Bessel function of first order. For the second moment of the stationary distribution, Eq. (20), we find

$$\langle N^2 \rangle = -\partial^2\phi/\partial q^2 = jL^2/2u \tag{28}$$

If we introduce the kink density m and antikink density n per unit length, we find for the variance for large L

$$\langle m^2 \rangle - \langle m \rangle^2 = \langle n^2 \rangle - \langle n \rangle^2 = (1/2L)n_0 \tag{29}$$

where $\langle m \rangle = \langle n \rangle = \langle N \rangle / L$, and n_0 is the stationary density for the infinitely long sample

$$n_0 = (j/2u)^{1/2} = \lim_{L\to\infty} \langle n \rangle \tag{30}$$

Equation (29) is a well-corroborated result in electron–hole statistics in semiconductors.[13,14] The connection of our problem to this semiconductor topic will become even more apparent in the following sections.

The rate at which $\langle \partial\theta/\partial t \rangle$ increases is of special interest.[1,2] Each kink and antikink passing a fixed point x along the ring brings an increase of 2π in $\theta(x)$. Therefore, we can express $\langle \partial\theta/\partial t \rangle$ in terms of the kink density and the velocity u as

$$\langle \partial\theta/\partial t \rangle = 2\pi u(\langle n \rangle + \langle m \rangle) \tag{31}$$

According to Eq. (27) we find

$$\langle \partial\theta/\partial t \rangle = 2\pi(2ju)^{1/2} I_1(x)/I_0(x) \tag{32}$$

Two limits are of interest. If the ring is short (the kink velocity u high) a kink pair traverses the ring and annihilates before the next pair is created.

In this case the growth rate is simply determined by the nucleation rate j. We find in this limit ($jL^2/2u \ll 1$)

$$\langle \partial\theta/\partial t\rangle = 2\pi jL\left(1 - \tfrac{1}{2}jL^2/2u\right) + \cdots \tag{33}$$

In the opposite limit ($jL^2/2u \gg 1$) we find

$$\langle \partial\theta/\partial t\rangle = 2\pi(2ju)^{1/2}\left[1 - (1/2L)(u/2j)^{1/2}\right] + \cdots \tag{34}$$

In lowest order both results (33) and (34) are in agreement with results from the theory of one-dimensional crystal growth,[10,11] and confirm results derived via simple balance equations.[2,11] Note that the length dependence of $\langle \partial\theta/\partial t\rangle$ is of importance if one wishes to determine activation energies through the measurement of the growth rate.[10,11] Because $j \sim e^{-\Delta E_N/kT}$ we find $\langle \partial\theta/\partial t\rangle \sim e^{-\Delta E_N/kT}$ in the limit $jL^2/2u \ll 1$, whereas in the limit $jL^2/2u \gg 1$ we obtain $\langle \partial\theta/\partial t\rangle \sim e^{-\Delta E_N/2kT}$. Here ΔE_N is the activation energy barrier[2] for the nucleation of a kink–antikink pair.

3. HYDRODYNAMICS OF THE DRIVEN KINK GAS

The configuration-space master equation discussed in the previous section describes the kink dynamics on a length scale large compared to the kink width. For phenomena occurring on a length scale large compared to the average distance between kinks, it is more convenient to go over to a hydrodynamic description, averaging the microscopic quantities over an interval ΔR which is small on the macroscopic length, but still contains many kinks. Therefore, we introduce the local densities of the kinks and antikinks

$$m(x,t) = \frac{1}{\Delta R}\int_{\Delta R} dx \sum_i \delta[x - x_i(t)] \tag{35}$$

$$n(x,t) = \frac{1}{\Delta R}\int_{\Delta R} dx \sum_i \delta[x - y_i(t)] \tag{36}$$

respectively, as hydrodynamic variables.

3.1. Derivation of the Hydrodynamic Equations

For hydrodynamic states the ensemble averages of the hydrodynamic densities $m(x,t), n(x,t)$ agree with the ensemble averages of the microscopic densities,

$$\begin{aligned}\langle m(x,t)\rangle &= \sum_{N=0}^{\infty}\int_{\Omega_N} dx^N dy^N \sum_i \delta[x - x_i(t)]\, p(C_N,t)\\ &= \sum_{N=0}^{\infty} N\langle \delta(x - x_1)\rangle_N p_N\end{aligned} \tag{37}$$

and similarly for $\langle n(x,t)\rangle$, where we have used (7). The subscript N in the far right-hand side of Eq. (37) denotes the average with respect to the N-particle probability $p(C_N, t)$. After some calculation we find from (11) the equations of motion for the averages

$$\partial\langle m\rangle/\partial t - u\partial\langle m\rangle/\partial x = j - 2u\langle mn\rangle \tag{38}$$

$$\partial\langle n\rangle/\partial t + u\,\partial\langle n\rangle/\partial x = j - 2u\langle mn\rangle \tag{39}$$

In the stationary and spatially uniform case we have $\langle mn\rangle = \langle n^2\rangle = j/2u$ in accordance with Eq. (28).

3.2. Fluctuations

If we rewrite Eqs. (38) and (39) as Langevin-type equations for the local densities m and n instead of their ensemble averages, we have to supplement these equations with stochastic forces ϕ_m and ϕ_n to allow for the fact that individual ensemble members can deviate from the ensemble average,

$$\partial m/\partial t - u\,\partial m/\partial x = j - 2unm + \phi_m \tag{40}$$

$$\partial n/\partial t + u\,\partial n/\partial x = j - 2unm + \phi_n \tag{41}$$

with $\langle\phi_m\rangle = \langle\phi_n\rangle = 0$. Similar equations have been studied by Brailsford[15] in connection with the theory of kinks in edge dislocations, and are also used to describe the electron–hole kinetics in semiconductors.[16]

The stochastic forces ϕ_m, ϕ_n have two sources. One of these is the diffusive Brownian motion of the kinks away from the deterministic path, which was neglected in (11) and will also be neglected in this section. This is a good approximation as long as the distance l_D, that a kink would diffuse during its lifetime $\tau = (2un_0)^{-1}$ is much smaller than the distance $l_u = u\tau = (2n_0)^{-1}$ that it travels during this time τ. With the diffusion constant $D \cong \mu kT$ of the kinks, where μ is their mobility,[2] we find $l_D \cong (D\tau)^{1/2}$. In the Ohmic limit, $u = \mu F$, and the ratio $l_D/l_u \cong (n_0 kT/F)^{1/2}$ is small[2] if $F \gg n_0 kT$. If the diffusive motion of the kinks is neglected, the remaining noise arises entirely from the nucleation and recombination processes. In this case ϕ_m and ϕ_n are totally correlated, $\phi_m(x,t) = \phi_n(x,t) = \phi(x,t)$. According to Section 2, the nucleation rate $(\partial m/\partial t)^{\text{nuc}} = (\partial n/\partial t)^{\text{nuc}}$ consists of a sequence of statistically independent nucleation events $\Phi_i(x,t) = \delta(x - x_i)\delta(t - t_i)$ which are randomly distributed in space and time. For (x,t) in the space–time interval $\Delta R\,\Delta T$ one has

$$\Phi^{\text{nuc}}(x,t) = \sum_{i=1}^{\Delta k} \Phi_i(x,t) = \sum_{i=1}^{\Delta k} \delta(x - x_i)\delta(t - t_i) \tag{42}$$

where the events (x_i, t_i) are uniformly distributed in $\Delta R\,\Delta T$, and their number Δk is Poisson distributed[17] with mean $\langle\Delta k\rangle = j\,\Delta R\,\Delta T$. The average is

$$\langle \Phi^{\text{nuc}}(x,t)\rangle = \langle \sum_i \Phi_i(x,t)\rangle = \frac{1}{\Delta R\,\Delta T}\langle \Delta k\rangle = j \tag{43}$$

as required, and the correlation function is

$$\begin{aligned}\langle \Phi^{\text{nuc}}(x,t)\Phi^{\text{nuc}}(x',t')\rangle &= \langle \sum_i \Phi_i(x,t)\Phi_i(x',t')\rangle + \langle \sum_{i\neq j} \Phi_i(x,t)\Phi_j(x,t')\rangle \\ &= \frac{1}{\Delta R\,\Delta T}\langle \Delta k\rangle \delta(x-x')\delta(t-t') \\ &\quad + \frac{1}{(\Delta R\,\Delta T)^2}\langle \Delta k(\Delta k - 1)\rangle \end{aligned} \tag{44}$$

if (x,t) and (x',t') are in the same interval $\Delta R\,\Delta T$, and

$$\langle \Phi^{\text{nuc}}(x,t)\Phi^{\text{nuc}}(x',t')\rangle = \frac{1}{(\Delta R\,\Delta T)^2}\langle \Delta k\rangle\langle \Delta k'\rangle \tag{45}$$

if they are in different intervals. Since $\langle \Delta k(\Delta k-1)\rangle = \langle \Delta k\rangle^2$, both cases are represented by

$$\langle \Phi^{\text{nuc}}(x,t)\Phi^{\text{nuc}}(x',t')\rangle = j\,\delta(x-x')\,\delta(t-t') + j^2 \tag{46}$$

and one obtains for the correlation of the fluctuations $\phi^{\text{nuc}} = \Phi^{\text{nuc}}(x,t) - j$

$$\langle \phi^{\text{nuc}}(x,t)\phi^{\text{nuc}}(x',t')\rangle = j\,\delta(x-x')\,\delta(t-t') \tag{47}$$

Consideration of the recombination rate $\Phi^{\text{rec}}(x,t) = (\partial m/\partial t)^{\text{rec}} = (\partial n/\partial t)^{\text{rec}}$ is similar but a little more subtle. First of all the recombination rate depends on the local densities of kinks and antikinks, and depends, therefore, in general on x and t. Additionally, while the nucleation process is stochastic, the recombination process is not—it occurs when kinks and antikinks are driven into each other. We will study the fluctuations of small deviations of the densities $\delta m(x,t) = m(x,t) - n_0$ and $\delta n(x,t) = n(x,t) - n_0$ from the uniform stationary state in *linear* response (random phase approximation[18]) to the stochastic force $\phi_m = \phi_n = \phi$. In this approach it is sufficient to consider the stochastic force ϕ which characterizes the unperturbed steady state. In Section 2 we have shown that in the steady state the kinks and antikinks are uncorrelated, and therefore the recombination events are random in space and time. For (x,t) in the time interval $\Delta R\,\Delta T$ one has

$$\Phi^{\text{rec}} = -\sum_{i=1}^{\Delta l} \delta(x-x_i)\delta(t-t_i) \tag{48}$$

Again Δl is Poisson distributed with mean $\langle \Delta l\rangle/\Delta R \Delta T = j = 2un_0^2$. One obtains therefore on the average

$$\langle \Phi^{\text{rec}}(x,t)\rangle = -j \tag{49}$$

and the correlation

$$\langle \Phi^{\text{rec}}(x,t)\Phi^{\text{rec}}(x',t')\rangle = j\,\delta(x-x')\delta(t-t') + j^2 \tag{50}$$

and hence for the fluctuations $\phi^{\text{rec}}(x,t) = \Phi^{\text{rec}}(x,t) + j$

$$\langle \phi^{\text{rec}}(x,t)\phi^{\text{rec}}(x',t')\rangle = j\,\delta(x-x')\,\delta(t-t') \tag{51}$$

Since the nucleation and recombination processes are statistically independent, one finds for the total fluctuation

$$\phi(x,t) = \phi^{\text{nuc}}(x,t) + \phi^{\text{rec}}(x,t) \tag{52}$$

the correlation function

$$\langle \phi(x,t)\phi(x',t')\rangle = 2j\,\delta(x-x')\delta(t-t') \tag{53}$$

3.3. The Normal Modes of the Kink Gas

To study the fluctuations of the kink gas it is convenient to discuss first the normal modes of Eqs. (40) and (41) describing the deviations from the stationary uniform state $m_0 = n_0 = (j/2u)^{1/2}$. We linearize Eqs. (40) and (41) with respect to small perturbations $\delta n, \delta m$ from the stationary state, *neglecting the stochastic force* ϕ. In view of the translational invariance in time and space, the perturbations can be taken to be of the form

$$\delta m(x,t) = \delta m_{q\omega} e^{iqx - i\omega t}, \qquad \delta n(x,t) = \delta n_{q\omega} e^{iqx - i\omega t} \tag{54}$$

The resulting eigenvalue problem leads to the characteristic equation $\Delta(q, \omega) = 0$, where

$$\Delta(q,\omega) = -\omega^2 - 4iun_0\omega + u^2q^2 \tag{55}$$

The dispersion relation $\omega(q)$ determined by $\Delta(q,\omega) = 0$ is

$$\omega_{1,2}(q) = -2in_0u \pm \left(u^2q^2 - 4n_0^2u^2\right)^{1/2} \tag{56}$$

For $q \ll 2n_0$, i.e., in the whole range in which the hydrodynamic description is valid, we find a branch of diffusive modes

$$\omega_1(q) = -iD_\rho q^2, \qquad D_\rho = u/4n_0 \tag{57}$$

and a branch of relaxation-type modes

$$\omega_2(q) = -4in_0u + iD_\rho q^2 \tag{58}$$

The slow branch of modes (57) arises because the difference between the numbers of kinks and antikinks obeys a conservation law giving rise to a continuity equation

$$\partial\rho/\partial t + \operatorname{div} j_\rho = 0, \qquad j_\rho = -u(m+n) \tag{59}$$

found by subtracting Eq. (41) from (40). Here, $\rho = m - n$ is the difference of the kink and antikink densities. The long-wavelength perturbations in ρ can decay only via diffusion, with diffusion constant $D_\rho = u/4n_0$. The second branch of modes (58) exhibits a nonvanishing relaxation rate, even at $q = 0$, and describes, in the long-wavelength limit, the behavior of perturbations of the total local kink density $\eta = m + n$.

The densities ρ and η are directly related to the sine-Gordon field $\theta(x, t)$ averaged over a macroscopically small interval ΔR in the same way as (35) and (36). Denoting this coarse-grained sine-Gordon field in this section also by $\theta(x, t)$, one has

$$\partial\theta(x,t)/\partial x = 2\pi(m - n) = 2\pi\rho \tag{60}$$

and [compare also Eq. (31)]

$$\partial\theta(x,t)/\partial t = 2\pi u(m + n) = 2\pi u\eta \tag{61}$$

where the conservation law (59) guarantees the compatibility of these equations. For later use note that our hydrodynamic equations [(40), (41)] are equivalent to the equation

$$\left(\frac{\partial^2\theta}{\partial t^2} - u^2\frac{\partial^2\theta}{\partial x^2}\right) + \frac{1}{2\pi}\left[\left(\frac{\partial\theta}{\partial t}\right)^2 - u^2\left(\frac{\partial\theta}{\partial x}\right)^2\right] = 4\pi u(j + \phi) \tag{62}$$

for the coarse-grained displacement field $\theta(x, t)$.

3.4. Correlation Functions of the Kink Gas

In linear response to the fluctuation force ϕ we obtain from Eqs. (40) and (41)

$$\delta m_{q\omega} = i\frac{\omega - qu}{\Delta(q,\omega)}\phi_{q\omega} \tag{63}$$

$$\delta n_{q\omega} = i\frac{\omega + qu}{\Delta(q,\omega)}\phi_{q\omega} \tag{64}$$

from which we can calculate the fluctuation spectra

$$\langle\delta m_{q\omega}\delta m^*_{q'\omega'}\rangle = S^{mm}(q,\omega)\,\delta(q - q')\,\delta(\omega - \omega') \tag{65}$$

and correspondingly for the other quantities. The fluctuation spectra for $\rho = n - m$ and $\eta = n + m$

$$S^{\rho\rho}(q,\omega) = \frac{8q^2u^2j}{\left(\omega^2 - q^2u^2\right)^2 + 16\omega^2u^2n_0^2} \tag{66}$$

$$S^{\eta\eta}(q,\omega) = \frac{8\omega^2 j}{\left(\omega^2 - q^2u^2\right)^2 + 16\omega^2u^2n_0^2} \tag{67}$$

satisfy $\omega^2 S^{\rho\rho}(q,\omega) - q^2u^2S^{\eta\eta}(q,\omega) = 0$ as a result of the conservation law (59). For $q \ll n_0$ we obtain two Lorentzian-type contributions corresponding, respectively, to the two types of long-wavelength modes [Eqs. (57) and (58)]

$$S^{\rho\rho}(q,\omega) = u\left(\frac{q^2}{\omega^2 + D_\rho^2 q^4} - \frac{q^2}{\omega^2 + 16n_0^2u^2}\right) \tag{68}$$

The diffusion part has a width $2D_\rho q^2$ and a height $u/D_\rho^2 q^2$, giving rise to a weight (width times height) $2u/D_\rho = 8n_0$, whereas the relaxation part has width $8n_0u$ and negative weight $q^2/2n_0 \ll 8n_0$. For the total density fluctuations the spectrum becomes

$$S^{\eta\eta}(q,\omega) = -\frac{1}{u}\frac{D_\rho^2 q^4}{\omega^2 + D_\rho^2 q^4} + \frac{8j}{\omega^2 + 16n_0^2u^2} \tag{69}$$

with the weights $q^2/2n_0$ and $8n_0 \gg q^2/2n_0$. The qualitative behavior of this spectrum is shown in Fig. 3a. The fluctuation spectrum of the coarse-grained sine-Gordon field fluctuations $\delta\theta = \theta - \langle\theta\rangle$ is found from (67) with the relation (61), yielding $\delta\theta_{q\omega} = i(2\pi u/\omega)\delta\eta_{q\omega}$. Here $\langle\ \rangle$ denotes the ensemble average. Later in the paper we shall also refer to the space average, which will be denoted by $\langle\ \rangle_R$. We find a fluctuation spectrum

$$S^{\theta\theta}(q,\omega) = 4\pi^2u\left(\frac{1}{\omega^2 + D_\rho^2 q^4} - \frac{1}{\omega^2 + 16n_0^2u^2}\right) \tag{70}$$

with a weight $32\pi^2 n_0/q^2$ for the diffusion peak and $-2\pi^2/n_0$ for the relaxation part. The main contribution of the kinks to the peak at $\omega = 0$ is due to the diffusion modes, as shown in Fig. 3b. It should be noted that the two peaks of Eq. (70) exist in addition to other contributions to $S^{\theta\theta}$ arising from the small disturbances within a Peierls valley.

Previous calculations[19–22] of the dynamic structure factor of systems exhibiting solitons refer to the undriven case. In the θ^4 system[19–21] each kink has two antikinks as neighbors. This has the consequence that $\rho(x,t) = 0$ on a hydrodynamic scale. A comparison of the earlier results[19–21] with Eq. (70) is therefore not meaningful. The $\sin\theta$ system has been studied numerically by Schneider and Stoll.[22] They investigated the kink gas in the relativistic limit (compared to the velocity of sound) and found that the dominant contribution to the low-frequency and long-wavelength part of the dynamic structure factor is due to propagating hydrodynamic modes $\omega = vq$, in contrast to diffusive hydrodynamic modes $\omega = -iD_\rho q^2$ found by us in the overdamped case.

We will now study the mean square difference in displacement $\langle[g(x, x_0;t)]^2\rangle$ between two particles $x - x_0$ apart in a finite ring chain of length

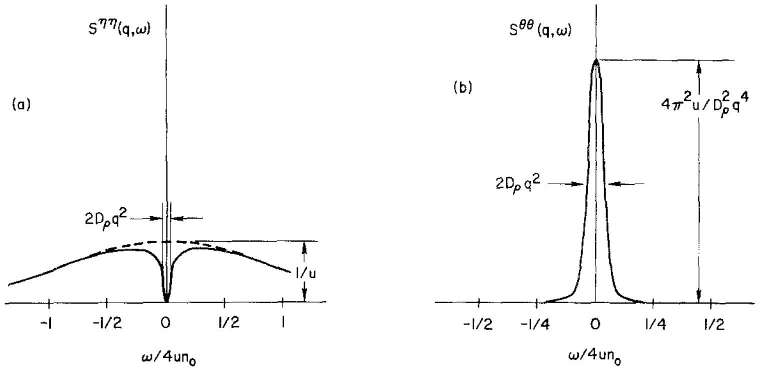

Fig. 3. (a) Low-frequency and long-wavelength fluctuation spectrum of the total kink and antikink density for fixed q as a function of $\omega/4un_0$. Here, $4un_0$ is the half-width of the relaxation part of the spectrum. (b) Low-frequency and long-wavelength fluctuation spectrum of the coarse-grained displacement field. (In both figures $q = n_0/2$ at the limit of validity of our approximations. At smaller q values, the diffusion part is too sharp to resolve in a casual illustration.)

L. The difference $g(x, x_0; t)$ is defined by

$$g(x, x_0; t) \equiv \theta(x,t) - \theta(x_0, t) \tag{71}$$

and may be expressed in terms of the Fourier components $\delta\theta_{q_n\omega}$ as

$$g(x, x_0; t) = (2\pi L)^{-1/2} \sum_n \int d\omega\, \delta\theta_{q_n\omega} e^{-i\omega t}(e^{iq_n x} - e^{iq_n x_0}) \tag{72}$$

where $q_n = (2\pi/L)n$, and the summation extends over all $n \neq 0$. This yields

$$\langle [g(x, x_0; t)]^2 \rangle = \frac{2}{\pi L} \sum_{n\neq 0} \int d\omega\, S^{\theta\theta}(q_n, \omega) \sin^2 \frac{q_n}{2}(x - x_0) \tag{73}$$

For large x it is sufficient to take the first term of Eq. (68) into account. Because of the translational invariance of the system, $\langle [g(x, x_0; t)]^2 \rangle$ depends only on the separation $x - x_0$ of the two particles. For convenience, we chose $x_0 = 0$ ($= L$). After integration over ω one finds

$$\langle [g(x, 0; t)]^2 \rangle = 4L \frac{u}{D_\rho} \sum_{n\neq 0} \frac{1}{n^2} \sin^2\left[\pi\left(\frac{x}{L}\right)n\right] \tag{74}$$

which yields

$$\langle [g(x, 0; t)]^2 \rangle = (8\pi^2/L) n_0 x (L - x) \tag{75}$$

where we have used $D_\rho = u/4n_0$ as defined in Eq. (57). Thus $\langle g^2 \rangle$ remains bounded for a finite sample. The maximum value occurs at $x = L/2$, where we find $\langle g^2 \rangle = 2\pi^2 n_0 L$. In the limit of an infinite sample ($L \to \infty$, x fixed)

Eq. (75) becomes

$$\langle [g(x,0;t)]^2 \rangle = 8\pi^2 n_0 x \tag{76}$$

The results (75), (76) will be rederived in the next section by a purely combinatorial approach.

Next we study the deviation of a particle from the center of gravity of its chain and from the ensemble average motion of the chain. This distinction is relevant only in a chain of finite length L, where the center of gravity exhibits diffusive motion around the ensemble average motion $\langle \theta(x,t) \rangle = 2\pi u \langle \eta \rangle t$ calculated in Ref. 2. The center of gravity of the chain is given by the spatial average $\langle \theta(x,t) \rangle_R = (1/L)\int \theta(x,t)\,dx$ of the displacement field. To find the deviation of the displacement field from the center of gravity $h_R(x,t) = \theta(x,t) - \langle \theta(x,t) \rangle_R$ we subtract the spatial average $\langle g(x,0;t) \rangle_R$ from $g(x,0;t)$. We find

$$h_R(x,t) = (2\pi L)^{-1/2} \sum_{n \neq 0} \int d\omega\, \delta\theta_{q_n \omega} e^{-i\omega t} e^{i q_n x} \tag{77}$$

and the mean square amplitude of the difference $\Delta h_R(x,t) = h_R(x,t) - h_R(x,0)$ is given by

$$\langle [\Delta h_R(x,t)]^2 \rangle = \frac{2}{\pi L} \sum_{n \neq 0} \int d\omega\, S^{\theta\theta}(q_n, \omega) \sin^2 \frac{\omega}{2} t \tag{78}$$

Using the small ω and q approximation (70) we find after integration over ω

$$\langle [\Delta h_R(x,t)]^2 \rangle = L \frac{u}{D_\rho} \left(\sum_{n \neq 0} \frac{1}{n^2} \left\{ 1 - \exp\left[-D_\rho \left(\frac{2\pi}{L} n \right)^2 t \right] \right\} \right) \tag{79}$$

and, therefore, in the limit $t \to \infty$

$$\Delta h_{\infty R}^2 = \lim_{t \to \infty} \langle [\Delta h_R(x,t)]^2 \rangle = \langle \langle [g(x,0;t)]^2 \rangle \rangle_R = \tfrac{4}{3} \pi^2 L n_0 \tag{80}$$

which is equal to the space average of (75). This equality can be understood in the following way. In the limit $t \to \infty$ the correlations between $h_R(t)$ and $h_R(0)$ are lost, and one finds $\Delta h_{\infty R}^2 = 2\langle h_{\infty R}^2 \rangle$. From the spectral decomposition of $\langle h_R^2 \rangle$ and $\langle g^2 \rangle$ it is seen that $\langle \langle g^2(x) \rangle \rangle_R = 2\langle h_{\infty R}^2 \rangle$, hence $\Delta h_{\infty R}^2 = \langle \langle g^2(x) \rangle \rangle_R$.

The deviation from the average motion (ensemble average) is

$$h_E(x,t) = \theta(x,t) - \langle \theta(x,t) \rangle \tag{81}$$

and the difference $\Delta h_E(x,t) = h_E(x,t) - h_E(x,0)$ is thus given by

$$\Delta h_E(x,t) = u(2\pi L)^{-1/2} \sum_n \int d\omega\, \delta\theta_{q_n \omega} e^{i q_n x} (e^{-i\omega t} - 1) \tag{82}$$

Note that the $q = 0$ mode is included, in contrast to Eq. (77). Therefore,

$$\langle [\Delta h_E(x,t)]^2 \rangle = \frac{2}{\pi L} \sum_n \int d\omega\, S^{\theta\theta}(q_n, \omega) \sin^2 \frac{\omega}{2} t \tag{83}$$

differs from Eq. (78) only in the $q = 0$ mode. Integration over ω yields

$$\langle[\Delta h_E(x,t)]^2\rangle = (4\pi^2 u/L)t + \langle[\Delta h_R(x,t)]^2\rangle \tag{84}$$

where we have also used the small ω and q approximation (70). The first term in Eq. (84) arises from the diffusive motion of the center of gravity and becomes dominant for times larger than $(L/2\pi)^2/D_\rho$.

In the infinite chain the center of gravity exhibits no diffusive motion and hence $\langle[\Delta h_R(x,t)]^2\rangle = \langle[\Delta h_E(x,t)]^2\rangle \equiv \langle[\Delta h(x,t)]^2\rangle$. For the infinite chain Eqs. (78) and (83) become

$$\langle[\Delta h(x,t)]^2\rangle = \frac{2}{\pi}\int d\omega\, S^{\theta\theta}(x=0,\omega)\sin^2\frac{\omega}{2}t \tag{85}$$

with

$$S^{\theta\theta}(x=0,\omega) = (1/2\pi)\int dq\, S^{\theta\theta}(q,\omega) \tag{86}$$

We find in the limit $\omega \ll un_0$

$$S^{\theta\theta}(x=0,\omega) = 2\sqrt{2}\,\pi^2(un_0)^{1/2}/\omega^{3/2} \tag{87}$$

and for $\omega \gg n_0 u$,

$$S^{\theta\theta}(x=0,\omega) = 8\pi^2(un_0)/\omega^2 \tag{88}$$

giving rise to the long-time behavior $(n_0 ut \gg 1)$

$$\langle[\Delta h(x,t)]^2\rangle = 8\pi^{3/2}(un_0)^{1/2}t^{1/2} \tag{89}$$

and a short-time behavior $(n_0 ut \ll 1)$

$$\langle[\Delta h(x,t)]^2\rangle = 8\pi^2(un_0)t \tag{90}$$

It should be noted that the linearization of the hydrodynamic equations (40) and (41) is not inconsistent with $h^2(x,t)$ becoming large, since the linearization concerns only the derivatives $\partial h/\partial t$ and $\partial h/\partial x$. The connection of our results to previous work[22–25] will be discussed at the end of the paper.

4. STEADY-STATE SPREAD OF A FINITE CHAIN IN THE DIRECTION OF PARTICLE MOTION

In this section we analyze the typical spread of a chain containing N kinks and N antikinks. As before, we assume periodic boundary conditions, $\theta(x+L) = \theta(x)$. The position θ of the chain at a point x relative to that at $x = 0$ depends on the number of kinks and antikinks between $x = 0$ and the point x [Eq. (71), $x_0 = 0$]. Neglecting small transient phonon displacements, the relative displacement is 2π multiplied by the difference between the number of kinks and antikinks in the interval $(0, x)$.

As has been shown in Section 2 for the driven case, all the possible distributions of kinks, for a given total number N are equally likely in the steady state [Eq. (4) and Section 2.5]. If instead of the driven case we consider the equilibrium case, we also find all configurations equally likely, since each configuration of N kinks has the same internal energy and is therefore associated with the same value of $e^{-\beta H}$. Whether the system is damped or not is irrelevant and does not affect the relative probability of various spatial patterns.

We are therefore dealing with a straightforward combinatorial problem. The probability that at x the chain is G kink steps ahead of the position at $x = 0$ is given by the probability of configurations that, out of a total of N kinks and N antikinks, place G more kinks than antikinks in the range $(0, x)$. This in turn can be found by evaluating the probabilities $p(r)$ for r kinks and $p(s)$ for $s = r - G$ antikinks in the given range, multiplying these two probabilities, and summing over all r. The probability $p(r)$ is given by the binomial distribution

$$p(r) = \binom{N}{r}\xi^r(1-\xi)^{N-r} \tag{91}$$

where $\xi = x/L$. For large N, $p(r)$ tends to a Gaussian distribution

$$p(r) = \left[2\pi N\xi(1-\xi)\right]^{-1/2}\exp\left\{-(r-N\xi)^2/\left[2N\xi(1-\xi)\right]\right\} \tag{92}$$

and therefore

$$P(G) = \int p(r)p(r-G)\,dr = \left[4\pi N\xi(1-\xi)\right]^{-1/2}\exp\left[-G^2/4N\xi(1-\xi)\right] \tag{93}$$

The variance of this distribution is given by $\langle G^2\rangle = 2N\xi(1-\xi)$. Noting that $g = 2\pi G$ and $N/L = n_0$, we find $\langle g^2(x)\rangle = 8\pi^2(n_0/L)x(L-x)$, which agrees with our result Eq. (75). For the derivation of Eq. (93) we have kept N fixed. Of course N is different for different ensemble members. For large N these variations become unimportant because the distribution function p_N [Eq. (20)] is peaked very sharply [see Eq. (29)].

Another derivation of Eq. (93) which is more complicated to describe, but is more instructive, treats this as a diffusion problem in which the x coordinate takes the place of time in an ordinary diffusion process. As x advances we at first have an equal probability for the occurrence of kinks and antikinks, and therefore diffuse away from $G = 0$. As we approach $x = L$, however, the total number of kinks and antikinks must become equal. The position G thus influences the relative probabilities of kinks and antikinks and pushes G back to zero. The largest excursion occurs at $x = L/2$. Equation (93) thus confirms and extends our earlier results derived via the hydrodynamic equations. Equation (93) is not limited to the driven case. The implication of these results will be discussed together with our previous results in the next section.

5. DISCUSSION

We shall here first briefly remind the reader of some of our key results and then proceed to discuss the time evolution of the spread in space of a sine-Gordon chain, comparing our results to those of others.

The equal probability for all spatial distributions of N driven kinks was the central result of Section 2. With this we were able to provide a more detailed derivation for the balance equations invoked in our earlier work.[2] The most significant result of the hydrodynamic approach is the existence of modes associated with a conserved quantity, the difference between the numbers of kinks and antikinks [Eqs. (57) and (59)]. This implies that the approach to the steady state is governed by slow modes decaying over a time proportional to L^2. *Therefore, experiments investigating properties of the system in the thermodynamic limit* $L \to \infty$ *have to be carried out over a very long time.*

The increase of the spatial dispersion of the chain with time has been discussed by several authors.[22–24] All of these authors consider the undriven system at equilibrium. The quantity investigated by these authors is $\langle[\Delta\theta]^2\rangle = \langle[\theta(x,t) - \theta(x,0)]^2\rangle$. Schneider and Stoll[22] followed an underdamped sine-Gordon chain by computer simulation and found that $\langle[\Delta\theta]^2\rangle \sim t^{4/3}$. It is not clear to us whether they intended this to be characteristic of a finite chain or of the thermodynamic limit. For independent-particle diffusion one would have $\langle[\Delta\theta]^2\rangle \sim t$, and the fact that they found motion which is faster than that deserves attention. By contrast, Imry and Gavish,[23] in much earlier work, studied the case in which the sinusoidal potential is absent and found $\langle[\Delta\theta]^2\rangle \sim t^{1/2}$ in the thermodynamic limit. Physically this result, indicating motion slower than diffusion, can be understood as follows. A particle initially can move fairly easily and essentially diffusively. It cannot, however, move too far away from its neighbors, and to move further, a longer segment of the chain must be strongly excited. Excitation of the longer segments is associated with longer time constants and leads to a slower diffusion. The motion of long segments of the chain is restrained by the stationary center of gravity of the infinite chain. This inhibits free diffusion. One might, therefore, expect similar qualitative behavior for the sine-Gordon chain in the case $V \neq 0$, i.e., one would expect the long-time behavior to be governed by an exponent smaller than 1.

In connection with the Schneider and Stoll simulation one may ask whether it was carried out for a long enough time to be significant or

Table I. Mean Square Fluctuations

	L infinite	L finite
Equilibrium	$\langle[\Delta\theta]^2\rangle \sim t^{1/2}$ (or t?) $\langle g^2\rangle \sim x$	$\langle[\Delta\theta]^2\rangle \sim t$ $\langle g^2\rangle \sim x[1-(x/L)]$ $\langle[\Delta h_R]^2\rangle < \infty$
Driven	$\langle[\Delta\theta]^2\rangle \sim t^2$ $\langle g^2\rangle \sim x$ $\langle[\Delta h]^2\rangle \sim t^{1/2}$	$\langle[\Delta\theta]^2\rangle \sim t^2$ $\langle g^2\rangle \sim x[1-(x/L)]$ $\langle[\Delta h_R]^2\rangle < \infty$ $\langle[\Delta h_E]^2\rangle \sim t$

whether their initial conditions were reasonable. Reference 25 stresses the importance of the initial conditions.

A recent analysis by Gunther and Imry[(24)] for an overdamped, undriven sine-Gordon chain yields $\langle[\Delta\theta]^2\rangle \sim t^{1/2}$ or t in the thermodynamic limit, depending on assumptions made by them. In view of the dependence on these further assumptions, the result of this analysis, while suggestive, cannot be taken as a completely definite answer.

The results of Refs. 23 and 24 are collected together with our results in Table I. We will now discuss the three quantities $\langle[\Delta\theta]^2\rangle$, $\langle[\Delta h]^2\rangle$, and $\langle g^2\rangle$ for each corner of the table separately. At equilibrium, in the thermodynamic limit, we include the results of Refs. 23 and 24 for $\langle[\Delta\theta]^2\rangle$ in the table. The behavior of $\langle g^2\rangle$ follows from the limit of Eq. (93) as $L\to\infty$.

In the undriven, finite chain the center of gravity exhibits normal diffusion, as we shall point out. There is no mechanism for long-term memory in our chain, even if underdamped. Particle velocities will change quickly, and on a longer time scale kinks will disappear and new ones will be nucleated. Thus if we choose a succession of very long time intervals we can be sure that the displacement of the center of gravity in each interval will be unrelated to the displacement in the preceding interval. That, in turn, assures us that the center of gravity, followed over a number of long intervals, will exhibit ordinary diffusion. A particular particle, followed over a sufficiently long period, must follow the center of gravity; Eq. (93) permits it to deviate only to a limited extent. Hence, if the particles diffuse along with the center of gravity we have $\langle[\Delta\theta]^2\rangle \sim t$, a result also found in Ref. 24. On the other hand, g, as defined in Eq. (71), measures the deviation of one particle with respect to another particle and h_R measures the deviation of one particle from the center of gravity of the chain. Therefore in $\langle g^2\rangle$ and $\langle[\Delta h]_R^2\rangle$ the motion of the center of gravity does not show up. We have shown in Section 4 that the chain has a limited spatial spread, which in turn determines values for $\langle g^2\rangle$ and $\langle[\Delta h_R]^2\rangle$.

For the driven, finite or infinite chain $\langle[\Delta\theta]^2\rangle$ is not a particularly interesting quantity, because it is simply governed by the average motion $\langle\theta(t)\rangle = \theta_0 + 4\pi u n_0 t$ as analyzed in Ref. 2, whence $\langle[\Delta\theta]^2\rangle \sim t^2$. What takes the place of $\langle[\Delta\theta]^2\rangle$ as an interesting quantity in the driven case is the mean square amplitude $\langle[\Delta h_E]^2\rangle$ of the displacement relative to the average motion.

For the finite, driven chain $\langle[\Delta h_R]^2\rangle$ has been shown to tend to a finite value as $t \to \infty$ [Eqs. (75) and (80)] and at long times $\langle[\Delta h_E]^2\rangle \sim t$ exhibits the diffusive motion of the center of gravity, Eq. (84).

For the infinite chain the behavior of $\langle g^2\rangle$ is given by Eq. (76), or the limit, as $L \to \infty$, of Eq. (93). In this case $\langle[\Delta h_R]^2\rangle = \langle[\Delta h_E]^2\rangle = \langle[\Delta h]^2\rangle$ and we have shown that $\langle[\Delta h]^2\rangle$ increases as $t^{1/2}$ for long times [Eq. (90)]. The fact that we find for the driven sine-Gordon chain the same result as that of Imry and Gavish[23] can be understood from the field equation (62) for the coarse-grained sine-Gordon field $\theta(x,t)$: if this equation is linearized around the average motion, $\theta(x,t) = \langle\theta(t)\rangle + \delta\theta(x,t)$, one finds

$$\frac{\partial^2\delta\theta}{\partial t^2} + 4un_0\frac{\partial\delta\theta}{\partial t} - u^2\frac{\partial^2\delta\theta}{\partial x^2} = 4\pi u\phi \qquad (94)$$

i.e., exactly the equation for a damped harmonic chain.

NOTE ADDED IN PROOF

Considerable progress has been made in the analysis of $\langle[\Delta\theta]^2\rangle$ for the infinite chain at equilibrium since the time of submission of this paper. Newer results are summarized by M. Büttiker and R. Landauer in *Physics in One Dimension*, J. Bernascoui and T. Schneider, eds. (Springer, Heidelberg), to be published.

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